Math 20550 - Calculus III - Summer 2014
July 16, 2014
Exam 3

Name: ____

There is no need to use calculators on this exam. This exam consists of 11 problems on 11 pages. You have 75 minutes to work on the exam. There are a total of 105 available points and a perfect score on the exam is 100 points. All electronic devices should be turned off and put away. The only things you are allowed to have are: a writing utensil(s) (pencil preferred), an eraser, and an exam. No notes, books, or any other kind of aid are allowed (except your notecard). All answers should be given as exact, closed form numbers as opposed to decimal approximations (i.e., π as opposed to 3.14159265358979...). **You must show all of your work to receive credit. Please box your final answers.** Cheating is strictly forbidden. Good luck!

Honor Pledge: As a member of the Notre Dame community, I will not participate in, nor tolorate academic dishonesty. My signature here binds me to the Notre Dame Honor Code:

Signature:____

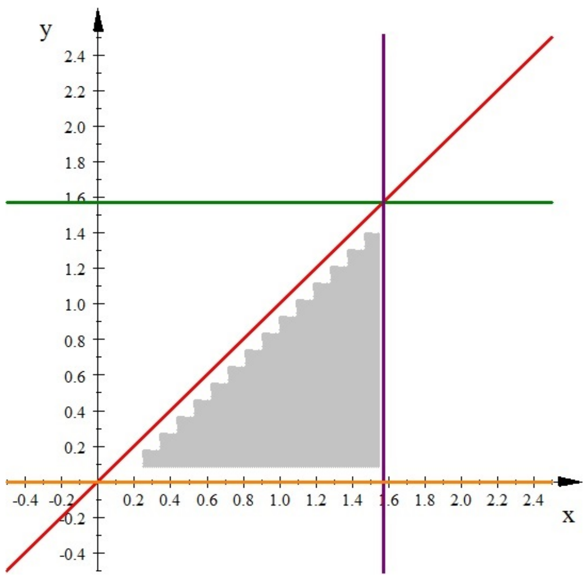
Problem	Score
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
9	/10
10	/5
11	/10
Score	/100

Problem 1 (10 points). *Compute the double integral*

$$\int_0^{\frac{\pi}{2}} \int_y^{\frac{\pi}{2}} \sin(x^2)dxdy$$

and sketch the region of integration.

Solution. The region is the shaded area here



The lines are: (red) $y = x$, (purple) $x = \frac{\pi}{2}$, (green) $y = \frac{\pi}{2}$, (orange) $y = 0$. Since the integral

$$\int \sin(x^2)dx$$

cannot be computed, we need to switch the order of integration. This gives

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \int_y^{\frac{\pi}{2}} \sin(x^2)dxdy &= \int_0^{\frac{\pi}{2}} \int_0^{x} \sin(x^2)dydx \\
&= \int_0^{\frac{\pi}{2}} x\sin(x^2)dx \overset{u=x^2}{=} \frac{1}{2}\int_0^{\frac{\pi^2}{4}} \frac{1}{2}\sin u\, du \\
&= -\frac{1}{2}\cos u\Big|_0^{\frac{\pi^2}{4}} = -\frac{1}{2}\cos\frac{\pi^2}{4} - \left(-\frac{1}{2}\cos 0\right) \\
&= \frac{1}{2}\left(1 - \cos\frac{\pi^2}{4}\right)
\end{aligned}$$

□

Problem 2 (10 points)**.**

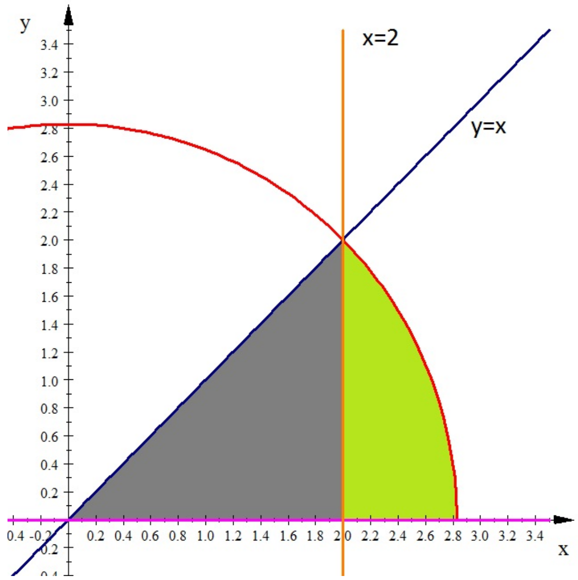
(a - 7 points) *Combine*

$$\int_0^2 \int_0^x \sqrt{x^2+y^2}dydx + \int_2^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}} \sqrt{x^2+y^2}dydx$$

into a single computable *integral. (*Hint*: sketch the region of integration)*

(b - 3 points) *Compute the integral in* (*a*)*.*

Solution. Plotting the two regions on the same plane gives



where the grey region is the region of integration from the first integral and the green region is from the second. Note that these two regions together give us the "first eighth" of the circle of radius $\sqrt{8} = 2\sqrt{2}$ (i.e., $0 \le \theta \le \frac{\pi}{4}$). Then, the integrals combine to, using polar coordinates:

$$\begin{aligned}
\int_0^2 \int_0^x \sqrt{x^2+y^2}dydx + \int_2^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}} \sqrt{x^2+y^2}dydx &= \int_0^{\frac{\pi}{4}} \int_0^{2\sqrt{2}} r(r\,drd\theta) \\
&= \int_0^{\frac{\pi}{4}} \int_0^{2\sqrt{2}} r^2\,drd\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{1}{3}r^3\Big|_0^{2\sqrt{2}} d\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{16\sqrt{2}}{3}d\theta = \frac{4\pi\sqrt{2}}{3}
\end{aligned}$$

$\square$

Problem 3 (10 points)**.** *Rewrite the triple integral*

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} xy\, dzdydx$$

using spherical coordinates. You do not need to compute it.

Solution. Let's begin by just paying attention to the z-integral. The lower bound is the cone $z = \sqrt{x^2+y^2}$ and the upper bound is the sphere $z = \sqrt{8-x^2-y^2}$ (so a sno-cone). Let's square both sides in both equations: $z^2 = x^2+y^2$ (cone) and $z^2 = 8-x^2-y^2$ (sphere). We find that their intersection is (by plugging one into the other) $z^2 = 8 - z^2 \Longrightarrow 2z^2 = 8 \Longrightarrow z^2 = 4 \Longrightarrow z = 2$. And when $z = 2$, we get a circle of radius 2: $x^2+y^2 = 4$. The shadow of this region in the xy-plane is the disk of radius 2. Looking now at the bounds of the outer two integrals, we see we really only want the piece of this disk in the first quadrant. Putting all this together, we want the piece of the sno-cone in the first octant.

Now, rewrite everything in spherical coordinates. The sphere is $\rho = 2\sqrt{2}$, the cone is $\phi = \frac{\pi}{4}$, and we know that $0 \le \theta \le \frac{\pi}{2}$. Thus, the rewritten integral is:

$$\begin{aligned}
\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} xy\, dzdydx &= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{2\sqrt{2}} (\rho\cos\theta\sin\phi)(\rho\sin\theta\sin\phi)\rho^2\sin\phi\, d\rho d\phi d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{2\sqrt{2}} \rho^4\cos\theta\sin\theta\sin^3\phi\, d\rho d\phi d\theta
\end{aligned}$$

□

Problem 4 (10 points)**.** *A thin spring has the shape of the helix*

$$x = t, y = \cos t, z = \sin t, 0 \leq t \leq 6\pi$$

and has linear density function $\rho(x, y, z) = x^2 + y^2 + z^2$. *Find the mass of the spring.*

Solution. The curve is

$$\mathbf{r}(t) = \langle t, \cos t, \sin t \rangle, 0 \leq t \leq 6\pi$$
$$\mathbf{r}'(t) = \langle 1, -\sin t, \cos t \rangle, \|\mathbf{r}'(t)\| = \sqrt{2}.$$

The mass is given by

$$\begin{aligned}
mass &= \int_C \rho(x, y, z)\, ds \\
&= \int_0^{6\pi} \rho(\mathbf{r}(t)) \|\mathbf{r}'(t)\| dt \\
&= \int_0^{6\pi} \left(t^2 + \cos^2 t + \sin^2 t\right) \sqrt{2} dt \\
&= \sqrt{2} \int_0^{6\pi} \left(t^2 + 1\right) dt \\
&= \sqrt{2} \left(\frac{1}{3}t^3 + t\right)\Big|_0^{6\pi} dt \\
&= \sqrt{2} \left(72\pi^3 + 6\pi\right)
\end{aligned}$$

□

Problem 5 (10 points)**.** *Compute* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where* $\mathbf{F} = (y^2z + 2xz^2)\mathbf{i} + 2xyz\mathbf{j} + (xy^2 + 2x^2z)\mathbf{k}$ *and* C *has parametric equations* $x = \sqrt{t}$, $y = t+1$, *and* $z = t^2$, $0 \le t \le 1$.

Solution. If you plug the curve into $\mathbf{F}$, you get something which doesn't look very pleasant. So, let's try and see if $\mathbf{F}$ is conservative. Then, if it is

$$\mathbf{F} = (y^2z + 2xz^2)\mathbf{i} + 2xyz\mathbf{j} + (xy^2 + 2x^2z)\mathbf{k} = \langle P, Q, R\rangle = \langle f_x, f_y, f_z\rangle$$

So

$$f = \int P\,dx = xy^2z + x^2z^2 + g(y,z)$$

Compare this to Q:

$$f_y = 2xyz + g_y(y,z) = Q = 2xyz$$

so

$$g_y(y,z) = 0 \quad\Longrightarrow\quad g(y,z) = h(z).$$

Now, we compare to R:

$$f_z = xy^2 + 2x^2z + h'(z) = R = xy^2 + 2x^2z$$

so

$$h'(z) = 0 \quad\Longrightarrow\quad h(z) = constant$$

Let's choose $h(z) = 0$, then

$$f = xy^2z + x^2z^2$$

is a potential for $\mathbf{F}$. Thus, we use the fundamental theorem of line integrals on this integral to get:

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C \nabla f\cdot d\mathbf{r}\\ &= f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(1,2,1) - f(0,1,0)\\ &= 5 - 0 = 5\end{aligned}$$

$\square$

Problem 6 (10 points)**.** *Consider the integral* $\iint_R \frac{x-2y}{3x-y} dA$, *where R is the parallelogram enclosed by the lines $x-2y=0$, $x-2y=4$, $3x-y=1$, and $3x-y=8$. Use a change of coordinates to rewrite the integral as a double integral over a rectangular region S. Sketch the region S. You do not have to compute the integral.*

Solution. Looking at the boundaries, choosing

$$u = x - 2y, \; v = 3x - y$$

seems like a good idea. The integrand becomes

$$\frac{x-2y}{3x-y} = \frac{u}{v},$$

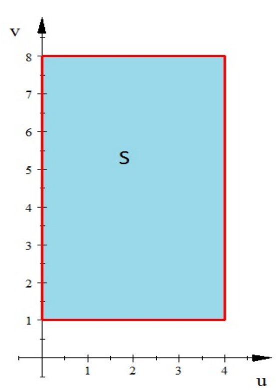
the Jacobian of this transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix}} = \frac{1}{-1-(-6)} = \frac{1}{5}$$

and the region S which maps to R under this transformation is

R:xy-plane	S:uv-plane
$x-2y=0$	$u=0$
$x-2y=4$	$u=4$
$3x-y=1$	$v=1$
$3x-y=8$	$v=8$

which a sketch of is



So the integral becomes

$$\iint_R \frac{x-2y}{3x-y} dA = \iint_S \frac{u}{v} \left|\frac{1}{5}\right| dudv = \int_1^8 \int_0^4 \frac{u}{5v} dudv$$

$\square$

Problem 7 (10 points)**.** *Compute the area inside the ellipse* $\frac{x^2}{4} + y^2 = 1$ *using an appropriate type of integral.*

Solution. There are at least two ways to do this. The easiest is to use Green's theorem, so that, if E denotes the ellipse

$$Area(E) = \iint_E dA = \frac{1}{2} \oint_{\partial E} x\,dy - y\,dx$$

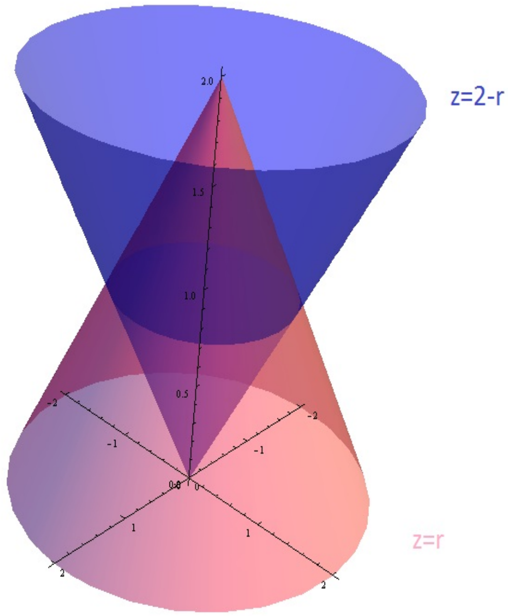
∂E has parametrization $\mathbf{r}(t) = \langle 2\cos t, \sin t \rangle, 0 \le t \le 2\pi$, which gives it the positive orientation. Then

$$\begin{aligned} A(E) &= \frac{1}{2} \int_0^{2\pi} [2\cos t\, d(\sin t) - \sin t\, d(2\cos t)] \\ &= \frac{1}{2} \int_0^{2\pi} \left[2\cos^2 t\, dt + 2\sin^2 t\, dt\right] \\ &= \int_0^{2\pi} dt = 2\pi \end{aligned}$$

$\square$

Problem 8 (10 points)**.** *Find the volume enclosed by the cones* $z = \sqrt{x^2 + y^2}$ *and* $2 - z = \sqrt{x^2 + y^2}$.

Solution. The solid bounded by the two cones is



This region is nicely described in cylindrical coordinates since then $z = r$ is the bottom cone and $z = 2 - r$ is the top cone. The shadow of this region in the xy-plane is $x^2 + y^2 \leq 1$ since the cones intersect at $z = r = 1$. Thus, the volume is

$$\begin{aligned} V(E) &= \iiint_E dV = \int_0^{2\pi} \int_0^1 \int_r^{2-r} r \, dz dr d\theta \\ &= \int_0^{2\pi} \int_0^1 (2r - r^2) dr d\theta = \int_0^{2\pi} \frac{1}{3} d\theta = \frac{2\pi}{3} \end{aligned}$$

$\square$

Problem 9 (10 points)**.** *Compute the divergence and curl of the vector field*

$$\mathbf{F} = xye^z\mathbf{i} + yze^x\mathbf{k}.$$

Solution.

$$\text{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xye^z) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(yze^x) = ye^z + ye^x$$

$$\text{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xye^z & 0 & yze^x \end{vmatrix} = \langle ze^x, xye^z - yze^x, -xe^z \rangle$$

□

Problem 10 (5 points)**.** *Suppose that* $\mathbf{F}$ *is a conservative vector field on* $\mathbb{R}^3$*, and that* $\mathbf{F}$ *is* C^1*. Compute the curl of* $\mathbf{F}$*.*

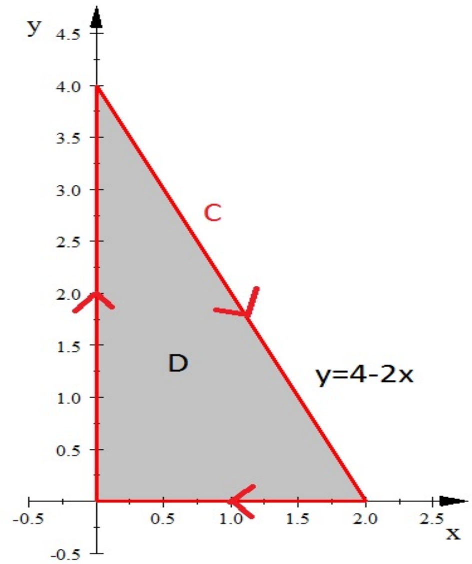
Solution. If $\mathbf{F}$ is conservative, then $\mathbf{F} = \nabla f = \langle f_x, f_y, f_z \rangle$, so

$$\text{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle = \langle 0, 0, 0 \rangle$$

□

Problem 11 (10 points). *Compute* $\int_C (y\cos x - xy\sin x)dx + (xy + x\cos x)dy$ *where* C *is the triangle traced out in moving from* $(0,0)$ *to* $(0,4)$ *to* $(2,0)$ *and back to* $(0,0)$.

Solution. A sketch of C:



Since the integrand looks bad, we will try Green's theorem. Also, notice the orientation is clockwise. This creates a minus sign in Green's theorem. So

$$\begin{aligned}
\int_C (y\cos x - xy\sin x)dx + (xy + x\cos x)dy &= -\iint_D \left[(y + \cos x - x\sin x) - (\cos x - x\sin x)\right] dA \\
&= -\iint_D y\, dA = -\int_0^2 \int_0^{4-2x} y\, dy dx \\
&= -\int_0^2 \frac{1}{2}y^2\Big|_0^{4-2x} dx = -\int_0^2 \frac{1}{2}(16 - 16x + 4x^2)dx \\
&= -\int_0^2 (8 - 8x + 2x^2)dx = -\left(8x - 4x^2 + \frac{2}{3}x^3\right)\Big|_0^2 \\
&= -\left(16 - 16 + \frac{16}{3}\right) \\
&= -\frac{16}{3}
\end{aligned}$$

□